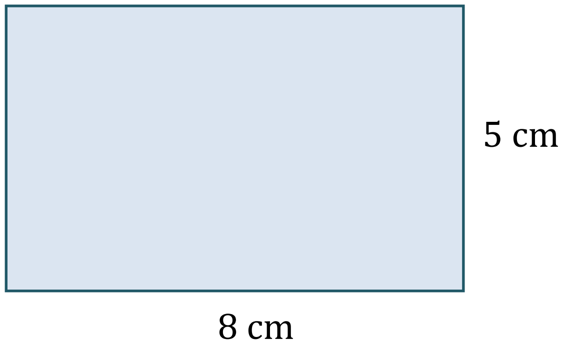
Stimulus question 90

Fibonacci Rectangles II

The Fibonacci sequence is the list of numbers starting 1, 1, 2, 3, 5, 8, 13, 21, …

The dimensions of a rectangle are measured to the nearest centimetre. They are two consecutive Fibonacci numbers: 5 cm and 8 cm.



The difference in the upper and lower bounds of the rectangle's area is 13 cm^2. It is noted that 13 is the next Fibonacci number after 5 and 8.

A conjecture is formed: if, when rounded to the nearest whole number, two sides of a rectangle are consecutive Fibonacci numbers, then the difference between the upper and lower bounds of the area of the rectangle is the next Fibonacci number.

It is true that if the problem is repeated for a rectangle with dimensions 8 and 13, then the difference between the bounds of area is 21.

Problem

Prove that this is always the case for a rectangle with side lengths that are two consecutive Fibonacci numbers.

Fibonacci Rectangles II scoring

Full credit

A complete proof; for example:

- Let a and b be two consecutive Fibonacci numbers that are the dimensions of a rectangle
- The bounds of a are $a - 0.5$ and $a + 0.5$
- The bounds of b are $b - 0.5$ and $b + 0.5$
- The lower bound of the area is $(a - 0.5)(b - 0.5) = ab - 0.5a - 0.5b + 0.25$
- The upper bound of the area is $(a + 0.5)(b + 0.5) = ab + 0.5a + 0.5b + 0.25$
- The difference between the bounds of the area is $(ab + 0.5a + 0.5b + 0.25) - (ab - 0.5a - 0.5b + 0.25) = 0.5a + 0.5a + 0.5b + 0.5b = a + b$
- Since a and b are two Fibonacci numbers then $a + b$ is the next Fibonacci number

Partial credit

- Sets up the conditions for an algebraic proof
- States the bounds of the dimensions algebraically
- States the lower bound of the area algebraically
- States the upper bound of the area algebraically
- States the difference between the bounds of the area algebraically
- Recognises the need to manipulate expressions in order to simplify the difference
- Manipulates the algebraic bounds by expanding brackets
- Simplifies the expression by collecting like terms
- Makes a conclusion

No credit

Any other response

Support
Set students the **Fibonacci Rectangles I** problem.

Stimulus question 89
Fibonacci Rectangles I

Suggest students try an algebraic approach

Draw a rectangle and label the sides *a* and *b*. Ask students to tell you what the shortest possible length (of either side) is if it has been rounded to the nearest integer.
Ask students to state the bounds of the integer length '*a*' if it has been rounded to the nearest whole number

Extension
Ask students if the dimensions of the rectangle have to be Fibonacci numbers for this to happen.
What is a **corollary** of this proof?

Fibonacci Rectangles II scoring (e)

Full credit
The sides need not be Fibonacci numbers for this to be the case.
In any rectangle where the sides have been given to the nearest whole number, the difference between the bounds of the area is the sum of the two dimensions of the rectangle.

Developing numerical reasoning

Identify processes and connections

- Select appropriate mathematics and techniques to use

Represent and communicate

- Explain results and procedures precisely using appropriate mathematical language
- **Generalise in words, and use algebra, to describe patterns that arise in numerical, spatial or practical situations**

Review

- **Justify numerical and algebraic results, making appropriate connections**