Contextualised task 24

Le Jeu de franc-carreau (The game of 'free tile')

Teaching notes

This task links probability and geometry within the context of a game that was commonly played in the seventeenth and eighteenth centuries. It is suitable for Intermediate tier, up until the extension task, which involves mathematics at Higher Tier.

Georges Louis Leclerc, Comte de Buffon, who analysed this game in 1733, was one of the first to make the link between probability and measuring.

The task consists of three parts:

1) To establish the probability of throwing a coin onto a square tile;
2) To make the game fair (by making a change to the coin);
3) Extension task (for Higher tier only): changing the size of the tile.

The resources for this task include:

- Teacher's script
- PowerPoint
- Sheets with 5 cm squares
- Mark scheme and solutions (solutions are also presented on slides 5-8 in the PowerPoint)
- Extension tasks and solutions

You will also need:

- Counters or coins. Be clear about the sizes of the counters you provide – probably the most useful would be 2 cm, but other sizes are available. If you provide the same size for the whole class, then they all do the same problem, but you could also provide different sized counters for different groups, so that they tackle variations of the same problem. This could lead to a greater focus on strategies, and comparisons of results.
- If you provide coins – probably plastic ones – then you will need to measure the diameters of each. The real 1p coins have a diameter of 20 mm, whereas the diameters of real 10p coins are 24 mm, and 2p coins are 26 mm in diameter.
- Rulers, pencils
- Calculators are a possibility (depending on the sizes of coins/counters used).

For the extension activity:

- Plain or squared paper – preferably centimetre squared paper, onto which learners can draw their own different-sized tiles.

Activity 1: Coins and tiles

This part of the task can be organised so that learners' work can be collected after about 15-20 minutes working on the problem, assessed with written feedback, and then returned for the next lesson. Then learners may reflect on the comments and try to improve their work. Good examples of working may be presented to the whole class for evaluation.

Teachers' script for PowerPoint presentation

The text in the right-hand boxes provides a possible script to be read to students. However, it is probably preferable to use your own words and elaboration. When questions are asked, time for discussion in pairs/groups should be provided. Ensure that students are given opportunity to explain their reasoning in response to these questions. All students need to understand the concepts in order to make progress with the task.

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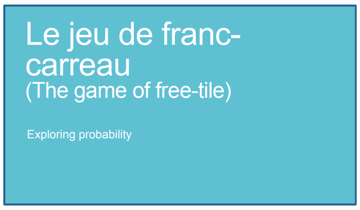
<tr>
<td>Slide 1</td>
<td>
</td>
<td>Keep this slide on the screen until you are ready to start the presentation. At this point, you may wish to hand out the 5 cm squares sheet, and the counters or coins.</td>
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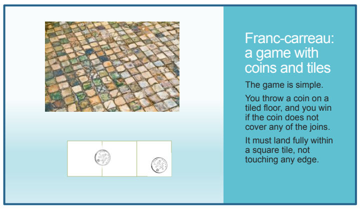
<tr>
<td>Slide 2</td>
<td>
</td>
<td>Read the slide for the class, or invite a member of the class to read it.
The coin on the left loses, the one on the right wins.

This is an old game played across Europe since the seventeenth and eighteenth centuries.
You have an old tiled floor like the one in the picture, and some coins. The coins are not French francs, but 'écus' that they had in those days.

The tiles in this picture are about 5 cm by 5 cm.</td>
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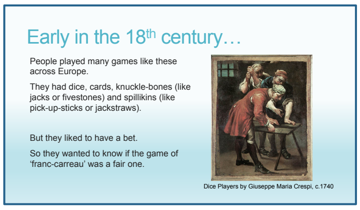
<td>Slide 3</td>
<td>
</td>
<td>Football and cricket were played in those days, but not with major teams.

In the early 1700s they knew how to find probabilities of rolling dice and of playing cards because they could identify all the possible outcomes and count them.</td>
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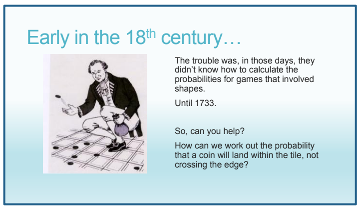
<td>Slide 4</td>
<td>
</td>
<td>However they didn't know how to find the probability that the coin would land fully inside the tile. Your task is to come up with a way of finding that probability.

You have counters (or coins) and a sheet of squares that represents tiles.

You may wish to invite learners to work in pairs or small groups. They can discuss some ideas about how they could find the probability.
Have a go at the problem, and we'll discuss your ideas after about 5 minutes (or suggest a deadline that will work with the class).
• What assumptions are you making?
• What questions do you have?
• What does the probability depend on?
• Can you write the probability in a general way?</td>
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</table>

		After some sharing of ideas, set a further deadline of about 5-10 minutes to enable learners to continue or complete their work. *At this point it may be useful to collect learners' work, in order to assess it and give feedback. This may be returned in the next lesson for them to improve their work. Alternatively, if different groups have explored both the theoretical and the experimental methods of finding probabilities, this may be the point where they can discuss their methods and results in class, inviting them to compare and evaluate.*

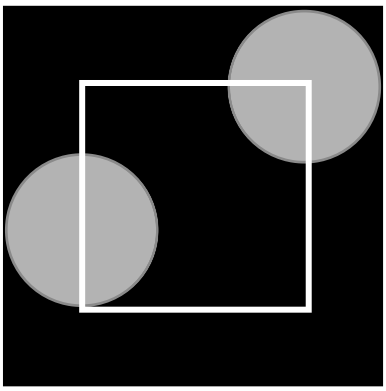
Commentary and solutions

It should be expected that learners either tackle this problem experimentally, by conducting a large number of trials and finding an estimate of the probability, or they consider a theoretical method. If different groups are given free choice, then the whole class may benefit from sharing the findings, and evaluating the different methods.

If several groups conduct experiments they could be encouraged to share their results, so that a better estimate of the probability is established from a large number of trials.

A theoretical solution might look like this, considering a 5 cm square tile, and a 1p coin (which has a diameter of 2 cm).

The winning area is a square (white outline) inside the tile (black) where the sides are a (silver) coin's diameter smaller than the sides of the tile. If the centre of the coin sits inside this area, then it cannot touch the sides of the tile. We then find the proportion of the area of the tile that the white square takes up.



The area of the tile is 25 cm^2.

The diameter of the coin is 2 cm, so the radius of the coin is 1 cm. There is a border around the white square whose width equals the radius of the coin: 1 cm.

So the side of the white square is 3 cm.

The area of the white square is 9 cm^2.

The probability of winning in this specific case is therefore $^9/_{25}$, or 36%.

The general case is:

$$\frac{(S - D)^2}{S^2} = P(\text{winning throw})$$

Where S is the side length of the tile, and D is the diameter of the coin.

This information is also presented on slides 5-8.

Mark Scheme

Full Credit

- If conducting an experiment to find the probability, states clearly that their solution is an *estimate*. Shows the number of trials they conducted, and explains why they have chosen that number. Gives their estimated probability in a suitable form.
- If calculating a theoretical probability, fully explains how it is calculated.
- Gives a general form for the probability, similar to the one given above. Symbols must be explained.
- Compares their solution with a solution from an alternative method, and if they are different, explains why.

Partial Credit

- Conducts a successful experiment to find a probability, correctly calculated and presented, but
    - Omits to state that this is an estimate, or
    - Omits to explain why they chose a particular number of trials
- Calculates correctly a theoretical probability for a specific case, but not a general formula.

No Credit

Any other response

Activity 2: How can we make the game fair?

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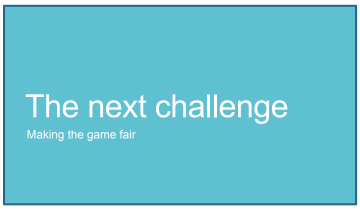
<td>Slide 9</td>
<td>
</td>
<td>Keep this slide on the screen until you are ready to start the presentation.

At this point, learners may still need copies of the 5 cm squares sheet, and the counters or coins, pencils and rulers.</td>
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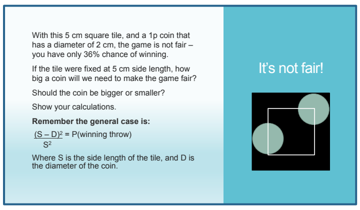
<tr>
<td>Slide 10</td>
<td>
</td>
<td>• So what is the probability that the coin will touch or cover the edges of the tiles? (We expect the answer 64%).

Here is the next challenge.
• What tells us that a game is fair?
• So what calculations do we need to make?</td>
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Commentary and solutions

This question invites learners to consider changing the size of the coin/counter when faced with a 5 cm tile in order to make the game fair. Extending the task further means it can become quite difficult quite quickly. With this question the task remains at Intermediate tier, but the later extension questions (changing the size of the tile rather than the coin, using a triangular grid) will use Higher tier mathematics.

To make the game fair the outcomes for winning and losing must be equal, and therefore the probability for each must be ½. The coin needs to be smaller to enlarge the winning area within the tile.

Using the given formula:

$$\frac{(S - D)^2}{S^2} = P(\text{winning throw})$$

Where S is the side length of the tile, and D is the diameter of the coin,

if S = 5 then:

$$\frac{(5 - D)^2}{5^2} = \frac{1}{2}$$

$(5 - D)^2 = \frac{1}{2}(25)$

$(5 - D)^2 = 12.5$

$(5 - D) = \sqrt{(12.5)}$

$5 - D = 3.5355$

$D = 5 - 3.5355$

D = 1.4645 cm = 1.5 cm rounded to 1 decimal place

Mark Scheme

Full Credit

- Correctly gives 1.5 cm as the solution.
- If calculating a theoretical probability, fully explains how it is calculated.
- Gives the answer to a suitable degree of accuracy, in this case to the nearest millimetre, as it is measurable.

Partial Credit

- Identifies that the probability of winning should be ½ and correctly sets up the formula using this.
- Gives an answer that is not rounded to a suitable degree of accuracy.

Limited Credit

Identifies that the probability of winning should be ½.
Makes an error in manipulating the formula.

No Credit

Any other response

Extension question

Offer this only to Higher tier students, as the quadratic formula comes into the solution.

If the coin were fixed at 2 cm diameter, how big should the tile be to make the game fair?

Solution:

Using the formula:

$\frac{(S - D)^2}{S^2}$ = P(winning throw)

$\frac{(S - 2)^2}{S^2} = \frac{1}{2}$

$2(S - 2)^2 = S^2$

$2(S^2 - 4S + 4) = S^2$

$2S^2 - 8S + 8 = S^2$

$S^2 - 8S + 8 = 0$

The quadratic formula now comes into play:

$S = (8 \pm \sqrt{32}) \div 2$

This leads to $4 \pm 2\sqrt{2}$, from which the only reasonable solution is 6.8 cm to 1d.p.

Progression in reasoning			
Identify processes and connections • **Consider connections between mathematical skills and contextualise these**	E.g. identifies that the probability in this game is linked to the area of the tile, compared to the area of an inner square. Identifies that the side of the inner square is smaller than the side of the tile by an amount equal to the diameter of the coin.	Establishes the area of the inner square as a fraction of the area of the tile, and uses this to correctly calculate the probability for winning. Uses this for a range of coins and tile sizes.	Identifies the need to use the quadratic formula in the extension task.
Represent and communicate • **Generalise in words, and use algebra, to describe patterns that arise in numerical, spatial or practical situations**	Explains in words how to calculate the probability, linking the diameter of the coin, the area of the tile and the area of the inner square.	Finds a general rule for finding the area of the inner square compared with the area of the tile, linking this to the diameter of the coin, and expresses this using symbols.	Applies the formula for the situation of making a fair game. Manipulates the formula effectively.
Review • **Justify numerical and algebraic results, making appropriate connections**	Explains in words and diagrams why the probability is linked to the proportion the area of inner square is compared with the area of the tile.	Shows the link between diagrams, words and symbols, how they establish a general rule for finding the probability of winning the game for any sized coin and square tile.	Uses the quadratic formula to establish which of the two solutions to use when finding the optimum size of tile and can explain their reasoning.

GCSE Content	
GCSE Mathematics – Numeracy and GCSE Mathematics	**GCSE Mathematics only**
Understanding number relationships and methods of calculation • Calculating using ratios in a variety of situations; proportional division.	
Understanding and using equations and formulae	• **Formation and manipulation of quadratic equations.** • **Solution by factorisation, graphical methods and formula, of quadratic equations of the form** $ax^2 + bx + c = 0$**, selecting the most appropriate method for the problem concerned.**
Understanding and using properties of position, movement and transformation • Solving problems in the context of tiling patterns and tessellation.	
Understanding and using measures • Calculating perimeter and area of a square.	
Estimating and calculating the probabilities of events • Understanding and using the vocabulary of probability, including notions of uncertainty and risk. • The terms 'fair', 'evens', 'certain', 'likely', 'unlikely' and 'impossible'.	• Use of: the probability of an event not occurring is one minus the probability that it occurs. (Probabilities must be expressed as fractions, decimals or percentages.) • Calculating theoretical probabilities. • Comparing an estimated probability from experimental results with a theoretical probability. • Knowledge that the total probability of all the possible outcomes of an experiment is 1.

Key

Foundation tier content is in standard text.
Intermediate tier content that is in addition to foundation tier content is in <u>underlined text</u>.
Higher tier content that is in addition to intermediate tier content is in **bold text**.